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# RANDOM PROCESSES

## DEFINITIONS

### RANDOM VARIABLE

Is a real valued function defined on the events of a probability space.

### CUMMULATIVE DISTRIBUTION FUNCTION

Is the probability of the random variable x being <=than a.

F(a) = P(x<=a)

F(a) is always positive, non decreasing, and right-continuous.

### DENSITY FUNCTION

Is a function defined on the random variable x, for which

f(x)>=0 and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\begin{cases} f(x_i) \geq 0 \\ \sum_i f(x_i) = 1 \end{cases}$$

The density function is the derivative of F(x).

$$F(a) = \int_{-\infty}^{a} f(x)\,dx \qquad \boxed{\frac{dF(x)}{dx} = f(x)} \;; \quad F(a) = \sum_{x_i \leq a} f(x_i)$$

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# EXPECTATION

Measure the probability of occurrence of an event.

$$E = \sum_i x_i \, p(x_i)$$

Expectation of a discrete function

$$E\{h(x_i)\} = \sum_i h(x_i) f(x_i)$$

Expectation of a continuous function

$$E\{h(x)\} = \int_{-\infty}^{\infty} h(x) f(x) \, dx$$

Properties: E{ } is a linear operator.

1. $E\{h_1(x) + h_2(x)\} = E\{h_1(x)\} + E\{h_2(x)\}$
2. $E\{C \cdot h(x)\} = C \cdot E\{h(x)\}$
3. $E\{C\} = C \quad ; C$
4. $E\{h(x)\} \geq 0 \quad , \text{ if } h(x) \geq 0$

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## MOMENTS

How do you characterize an arbitrary density function?

rth moment about the origin

$$\mu_r' = E\{x_i^r\} = \sum_i x_i^r f(x_i)$$

$$\mu_r' = E\{x^r\} = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

First moment is the MEAN.

$$\mu = \mu_1' = E\{x\} = \int_{-\infty}^{\infty} x f(x)\, dx$$

The rth moment around the means is

$$\mu_r = E\{(x_i - \mu)^r\} = \sum_i (x_i - \mu)^r f(x_i)$$

$$\mu_r = E\{(x-\mu)^r\} = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx$$

VARIANCE is the second moment around the mean.

$$\sigma^2 = \mu_2 = E\{(x-\mu)^2\} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

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# BIVARIATE DISTRIBUTION

Measures statistical relations between two random variables

Joint cummulative distribution function F(a1,a2)

$$F(a_1, a_2) = P(x_1 \leq a_1 \text{ and } x_2 \leq a_2)$$

Joint density function f(x1,x2)

$$\begin{cases} f(x_{i1}, x_{j2}) \geq 0 \\ \sum_{i_1} \sum_{j} f(x_{i1}, x_{j2}) = 1 \end{cases} \qquad \begin{cases} f(x_1, x_2) \geq 0 \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 \end{cases}$$

Expectation

$$E\{h(x_{i1}, x_{j2})\} = \sum_{x_1} \sum_{x_2} h(x_{i1}, x_{j2})\, f\underbrace{(x_{i1}, x_{j2})}$$

$$E\{h(x_1, x_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x_1, x_2)\, f(x_1, x_2)\, dx_1\, dx_2$$

Bivariate expectation is still linear operator.

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## INDEPENDENCE

Two r.v. are independent if

$$P(x_1 \le a_1 \text{ and } x_2 \le a_2) = P(x_1 \le a_1) \cdot P(x_2 \le a_2)$$

$$\Longrightarrow \begin{cases} F(a_1, a_2) = F(a_1) \cdot F(a_2) \\ f(x_1, x_2) = f(x_1) \cdot f(x_2) \end{cases}$$

If independent, then $\boxed{E\{h1(x1)h2(x2)\} = E\{h1(x1)\} \cdot E\{h2(x2)\}}$

Can also define moments. The new one is covariance (second mixed moment

$$\mu_{11} = \sigma_{x_1 x_2} = E\{(x_1 - \mu_1)(x_2 - \mu_2)\}$$

When independent, covariance is zero.

The normalized covariance is called the correlation coefficient

$$\rho_{x_1 x_2} = \frac{\sigma_{x_1 x_2}}{\sigma_{x_1} \cdot \sigma_{x_2}} \qquad \boxed{-1 \le \rho \le 1}$$

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# RANDOM OR STOCHASTIC PROCESS

Is a set of functions of some parameter (time), together with a probability measure. When parameter is time random processes are called STOCHASTIC processes.

Example: Mean squared value of thermal noise

$$v^2 = 4kTR\Delta f \quad (\text{volts})^2$$

k = BOLTZMAN CONST
T — ABS. TEMP.
R — RESIST $\Omega$
$\Delta f$ — MEASUREMENT BANDWIDTH

Each memeber of the set is a REALIZATION. At any time t1, the realization of v(t) is a realization of a random variable v(t1).

Therefore, a random process can also be defined as indexed family of random variables {x(t), t∈T}.

Depending upon the index set, r.p. can be discrete parameter or continuous parameter. The RANDOM variable ~~time~~ can also be discrete or continuous.

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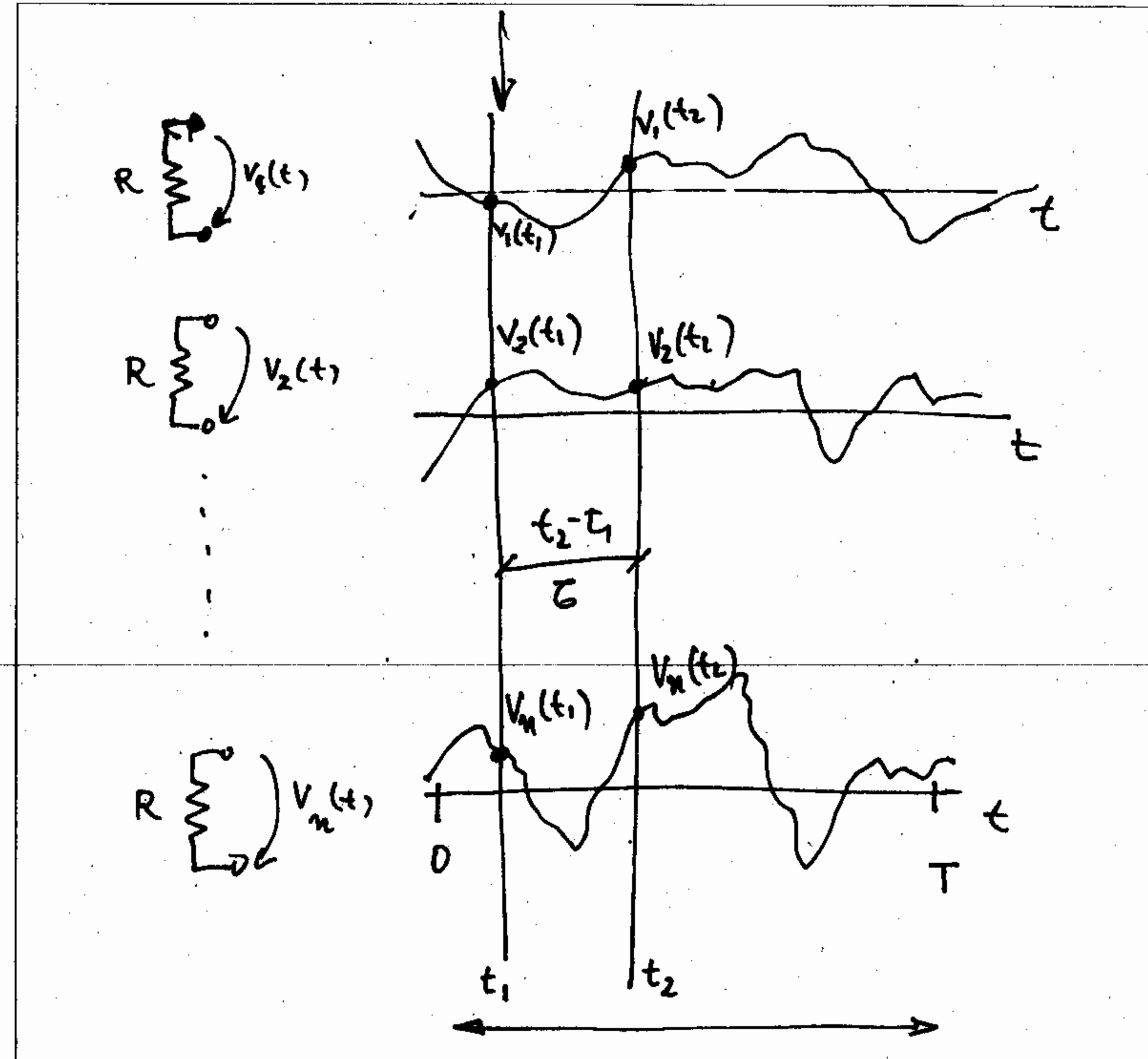

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# STATIONARITY

A r.p. is stationary if its statistical properties do not change with time.

If the properties do change with time, then the r.p. is called nonstationary. Stationarity conditions can be given in terms of the moments.

We can define the density function of the r.v. v(t1), f[v(t1)], which is called the 1st order amplitude probability density function.

A process is stationary to order one in (0,T) if

$$f(v(t_1)) = f(v(t_1+h)) \quad \forall\, \underline{h},\ t_1+h \in [0,T]$$

Therefore such processes have constant mean.

$$E\{v(t_1)\} = E\{v(t_1+h)\} = \underline{\mu}$$

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Now consider all joint distributions f2[v(t1),v(t2)]

$$f_2(v_1, v_2) = \text{PROB}\begin{cases} v(t_1) \leq v(t_1) \leq v(t_1) + d\,v(t_1) \\ v(t_2) \leq v(t_2) \leq v(t_2) + d\,v(t_2) \end{cases}$$

We define a process stationary of order two in (0,T) if

$$f_2(v(t_1), v(t_2)) = f_2(v(t_1+h), v(t_2+h))$$

The expected value is

$$E\{v(t_1).v(t_2)\} = E\{v(t_1+h).v(t_2+h)\} = E\{v_1 v_2\}$$

and it is called the <u>autocorrelation function</u>. For stationary processes of order two the autocorrelation function only depends on the time lag (t1-t2).

$$R(t_1,t_2) = E\{v(t_1).v(t_2)\} = R(t_1+h, t_2+h)$$

FOR $h = -t_1$

$$R(t_1,t_2) = \underline{R(t_2-t_1)}$$

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When E{v(t)} is zero , R(t1,t2) is called the covariance function.

One can define stationarity to order n in (0,T) in the same way

$$f_n\left[v(t_1),\dots v(t_n)\right] = f_n\left[v(t_1+h),\dots,v(t_n+h)\right]$$

A r.p. that stationary to all orders is called strictly stationary.

Normally we are only interested in the case n=2.,

A r.p. is called widesense stationary if

$$E\{|v(t_i)^2|\} < \infty \quad \forall t_i$$

$$E\{v(t_1).v(t_2)\} = R(t_2-t_1) \quad \forall t_1,t_2$$

It is often convenient to also require E{v(t)}=u= const.

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# PROPERTIES OF AUTOCORRELATION FUNCTIOn

$$t_2 - t_1 = \tau \qquad R(\tau) = E\{v(t) \cdot v(t+\tau)\}$$

- The mean square value is R(0).

$$R(0) = E\{(v(t))^2\}$$

- The autocorrelation function is even

$$R(\tau) = R(-\tau)$$

- The autocorrelation function is maximum at the origin

$$R(0) \geq |R(\tau)|$$

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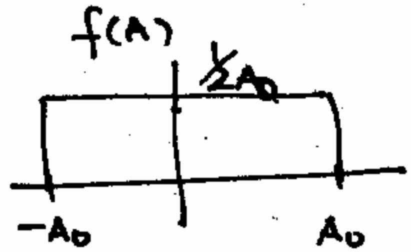
EXAMPLE:

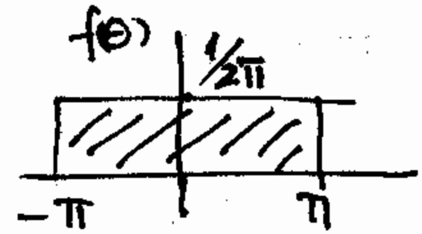
$$e(t) = A \sin(\omega t + \theta)$$

$A, \theta$, r.v.

$A \rightarrow$ UNIFORM OVER $\{-A_0, A_0\}$

$\theta \rightarrow$ " " $\{-\pi, \pi\}$.




CASE I; $\theta$ r.v. $\underline{A = \text{CONST.}}$ STATIONARY?

$$R(t,\tau) = R(t_1, t_2) = E\{e(t_1) \cdot e(t_2)\} = E\{e(t) \cdot e(t+\tau)\}$$

$$= E\{A^2 \sin(\omega t + \theta) \sin(\omega(t+\tau) + \theta)\}$$

$$\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

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$$R(t, t+\tau) = \frac{1}{2} E\left\{ A^2 \cos\omega\tau - A^2 \cos(2\omega t + 2\theta + \omega\tau) \right\}$$

$$= \frac{A^2}{2} \cos\omega\tau - \frac{A^2}{2} \cdot \underbrace{\int_{-\pi}^{\pi} \cos(2\omega t + 2\theta + \omega\tau) \underbrace{\frac{1}{2\pi}}_{f(\theta)} d\theta}_{\equiv 0}$$

So WIDESENSE STATIONARY

CASE II $\quad \underline{\underline{A}}$ r.v. $\quad \omega, \theta$ const.

$$R(t, t+\tau) = \frac{1}{2}\left[\cos\omega\tau - \cos(2\omega t + 2\theta + \omega\tau)\right] \underbrace{\int_{-A_0}^{A_0} A^2 \cdot \frac{1}{2A_0} dA}_{\frac{A_0^2}{3}}$$

NONSTATIONARY (DEPENDS ON $\underline{\underline{t}}$)

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# ERGODICITY

To obtain any of the previous measures one needs statistical averages, i.e.the whole r.p.. This may not be feasible.

The assumption of ergodicity allows us to replace statistical averages by time averages in one realization of the r.p.

A r.p. is ergodic if the time averages in any of the realizations are equal to the statistical averages.

Ergodicity implies stationarity.

In engineering we almost always assume ergodicity.

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# TIME AVERAGES

We define time average of r.p. v(t) as

$$A\{v(t)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} v(t)\,dt \qquad A\{v_n\} = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} v_n$$

The time autocorrelation function is

$$\mathcal{R}(\tau) = A\{v(t)\cdot v(t+\tau)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} v(t)\cdot v(t+\tau)\,dt$$

$$\mathcal{R}(m) = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{-N}^{N} v_n \cdot v_{n+m}$$

and the time crosscorrelation function

$$\mathcal{R}_{uv}(\tau) = A\{u(t)\cdot v(t)\}$$

For ergodic widesense stationary r.p.

$$E\{v(t)\cdot v(t+\tau)\} = A\{v(t)\cdot v(t+\tau)\}$$

$$\begin{cases} E\{v(t)\} = A\{v(t)\} = \eta \\ R(\tau) = \mathcal{R}(\tau) \end{cases}$$

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## CROSS-CORRELATION FUNCTION

Two r.p. u(t) and v(t) with autocorrelation functions $R_{uu}(t1,t2)$ and $R_{vv}(t1,t2)$. The cross-correlation function is

$$R_{uv}(t_1,t_2) = E\{u(t_1).v(t_2)\}$$

or

$$R_{vu}(t_1,t_2) = E\{v(t_1).u(t_2)\}$$

PROPERTIES of cross-correlation function.

- $R_{uv}(\tau) = R_{vu}(-\tau)$
- $|R_{uv}(\tau)| \leq \frac{1}{2}\left[R_{uu}(0) + R_{vv}(0)\right]$
- $|R_{uv}(\tau)|^2 \leq R_{uu}(0).R_{vv}(0)$

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## FOURIER TRANSFORMS

If one takes the Fourier transform of the r.p. x(t) we get

$$X(\omega)=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

X(w) is another r.p., now complex, with real parameter w.

If the r.p. x(t) is at least wide-sense stationary with constant mean of zero, we can define power spectral density as the Fourier transform of the autocorrelation function.

$$\begin{cases} \varphi(\omega)=\int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau \\ R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\omega)\, e^{j\omega\tau}\, d\omega \end{cases}$$

We can also show that the same result is valid for the time autocorrelation function

$$\varphi(\omega)=\int_{-\infty}^{\infty} \mathcal{R}(\tau)\, e^{-j\omega\tau}\, d\tau \quad ; \quad \mathcal{R}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\omega)\, e^{j\omega\tau}\, d\omega$$

and also that

$$\varphi(\omega)=\lim_{T\to\infty} E\left\{\frac{|X_T(\omega)|^2}{2T}\right\}$$

$$X_T(\omega)=\int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt$$

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## SPECTRUM OF DISCRETE TIME INFINITE ENERGY SIGNALS

Recall that the mean is defined

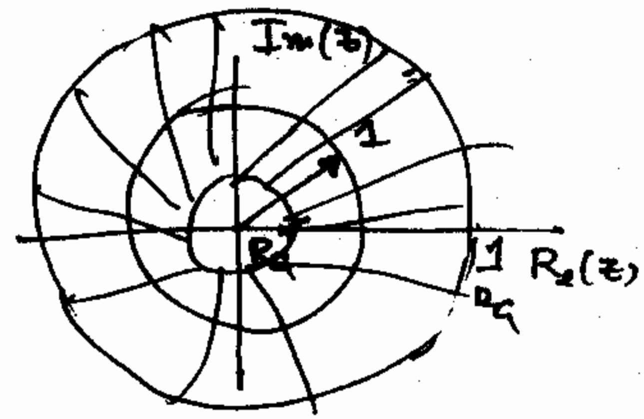
$$E\{v_n\} = \mu \quad ; \quad A\{v_n\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} v_n$$

and the autocorrelation function

$$R(m) = E\{v_n \cdot v_{n+m}\} \quad ; \quad C(m) = E\{(v_n-\mu)(v_{n+m}-\mu)\} = R(m) - \mu^2$$

Let us define the z tranform of the autocovariance function(or of the autocorrelation function for zero mean r.p.)

$$\begin{cases} \gamma_{vv}(z) = \sum_{n=-\infty}^{\infty} C(n) z^{-n} \\ C(n) = \frac{1}{2\pi j} \oint_c \gamma_{vv}(z) z^{n-1} dz \end{cases}$$




$$R_a < |z| < \frac{1}{R_a}$$

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PROPERTIES

1. The integral of the power spectral density is the average power of the r.p.

$$R(0) = E\{x^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\omega)\, d\omega$$

2. The power spectral density is real

$$\varphi(\omega) = 2\int_{0}^{\infty} R(\tau)\cos\omega\tau \, d\tau$$

3. The power spectral density is non negative.

4. the power spectral density is an even function

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This function has nice properties

- $$\sigma_v^2 = \frac{1}{2\pi j}\oint_c \gamma_{vv}(z)\, z^{-1}\, dz$$
- $$\gamma(z) = \gamma(1/z^*)$$ $$\gamma_{xv}(z) = \gamma_{vx}(1/z^*)$$

For discrete time signals the power density spectrum is defined as the z-tranform of the autocovariance function calculated on the unit circle

$$\gamma(e^{jw}) = \gamma(z)\Big|_{z=e^{jw}} = \sum_{n=-\infty}^{\infty} C(n)\, e^{-jwn}$$

CAN INTEGRATE ON W, i.e.

$$P(\omega) = \gamma(e^{jw}) \Rightarrow \qquad \sigma_v^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\omega)\, d\omega$$

The area under P (w) is proportional to the average power.

- $P(\omega) = P(-\omega)$
- $P(\omega)$ NON NEGATIVE

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# RESPONSE OF LINEAR SYSTEMS TO RANDOM SIGNALS

Consider a shift invariance linear system with constant coefficients

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\,x(k) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

If the input is stationary, then the output is also stationary. In fact the output mean is

$$m_y = E\{y(n)\} = \sum_k h(k)\,E\{x(n-k)\} = m_x \sum_k h(k) = H(e^{j0})\,m_x$$

and the output autocorrelation function is

$$R_y(n, n+m) = E\{y(n)\,y(n+m)\} = E\left\{\sum_k \sum_r h(k)h(r)\,x(n-k)\,x(n+m-r)\right\}$$

$$E\{x(n-k)\,x(n+m-r)\} \rightarrow R(m+k-r) \Rightarrow \sum_k h(k) \sum_r h(r)\,R(m+k-r)$$

In the frequency domain this result is much more readily interpreted.

$$\gamma_{yy}(z) = V(z)\,\gamma_{xx}(z)$$

$$V(z) = Z\left[\sum_k h(k)\,h(k+l)\right] = H(z)\,H(z^{-1})$$

OR $\quad P_{yy}(\omega) = |H(e^{j\omega})|^2\,P_{xx}(\omega)$

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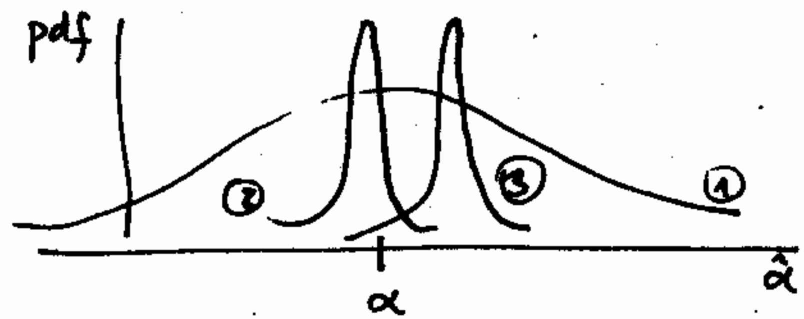
# ESTIMATION

The problem is that time autocorrelation function (or mean)can not be computed accurately (infinite time). We would like to now what ~~are~~ is the error that we make when we use finite time to compute the mean and time autocorrelation function. This problem is studied in spectral analysis, more particularly in estimation theory. Here just a review.

If we use

$$\hat{m}_x = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \qquad \text{(MAXIMUM LIKELIHOOD)}$$

to compute the time mean, if N sufficiently large, good estimate. However, $\hat{m}_x$ is a r. variable. So, can define its mean and variance (or its pdf). Different values of computing the mean will have different pdf. We would like one that



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So quality factors are the bias and the variance defined as

$$B = \text{bias} = \alpha - E[\hat{\alpha}]$$

UNBIASED ESTIMATOR $B = 0$

$$\sigma_{\hat{\alpha}}^2 = E\left[(\hat{\alpha} - E[\hat{\alpha}])^2\right]$$

If the bias and the variance go to zero as the number of observations increases, then the estimator is called consistent. Maximum likelihood estimators are normally used.

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Let us define the windowed autocorrelation function of x(n) as

$$C_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-|m|-1} x(n)\, x(n+m)$$

This function is a biased estimate of the true (infinite autocorrelation function)

$$E\left[C_{xx}(m)\right] = \frac{N-|m|}{N} R_{xx}(m) \implies B = R_{xx}(m)\left[\frac{m}{N}\right]$$

and the variance of the estimate also depends on N $\left(\propto \frac{1}{N}\right)$. However for large N the mean and the variance approach the true values of the autocorrelation function. So the windowed autocorrelation function is a consistent estimator of the autocorrelation function.

We could think that the Fourier transform of the windowed autocorrelation function was also an asymptotically unbiased estimator of the power spectral density. Unfortunately this is not the case.

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The Fourier transform of $c_{xx}(n)$ is called the PERIODOGRAM

$$I_N(\omega) = \sum_{m=-(N-1)}^{N-1} c_{xx}(m) e^{-j\omega m}$$

It turns out that it can also be computed as

$$I_N(\omega) = \frac{1}{N} \left| X(e^{j\omega}) \right|^2$$

$$X(e^{j\omega}) = \sum_{0}^{N-1} x(n) e^{-j\omega n}$$

This formula tells us that one way to efficiently compute the periodogram is with the FFT (just square it and take the absolute value).

However the periodogram is not a consistent estimator of the power spectrum (i.e. it may show peaks that do not correspond to resonances in the real data). However there are ways to decrease the varaince through smoothing (Welch, etc).

~~RESPONSE OF LINEAR SYSTEMS TO RANDOM SIGNALS~~

~~Consider a shift invariance linear system with constant coefficients~~