Luis

Wiener Filters

widesense

$x[n] :=$ stationary random process

$E[x[n]] =$ constant

$E[x[i] x[i]]$

$E[x[\quad n-k]] = f(i-k)$

$u[n] = x[n] + n[n]$

$L[x[n]] = d[n]$: desired signal

Goal Find a linear system that takes

$u[n] \longmapsto y[n]$, such that $y[n]$ is close to $d[n]$

$$J = E[\varepsilon^2[n]] = E[(d[n] - y[n])^2]$$

$$y[n] = h[n] * u[n]$$

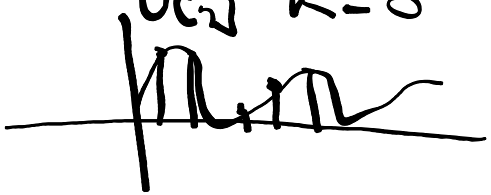
our linear sytem is time invariant

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k]}$$

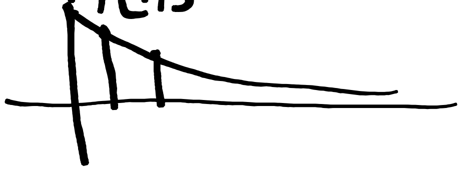
find $h[k]$ such that J is minimized

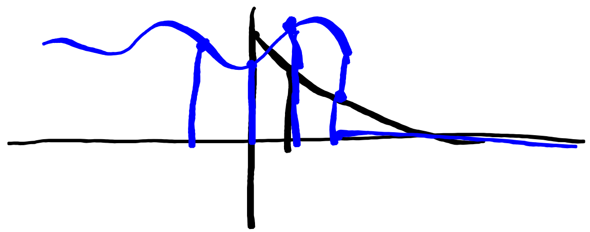
$$y[n] = \sum^{\infty} h[k]\, u[n-k]$$

$u[n]$ $k=0$



$h[n]$



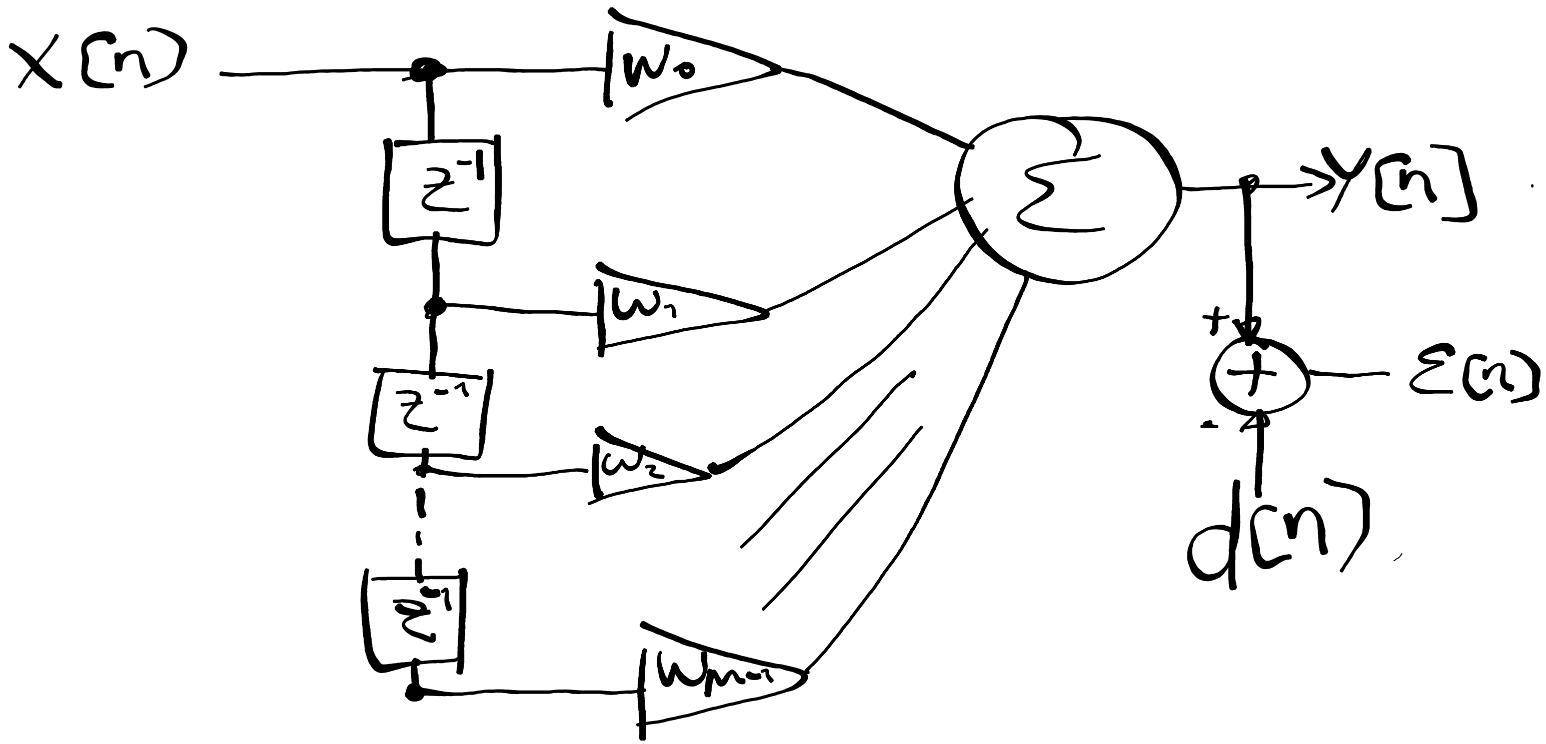


FIR finite impulse response

$$y[n] = \sum_{k=0}^{M-1} h[k]\, u[n-k] = \sum_{k=0}^{M-1} w_k\, u[n-k]$$

- pure prediction $n[n] = 0$
- prediction + estimation $n[n] \neq 0$

our scope

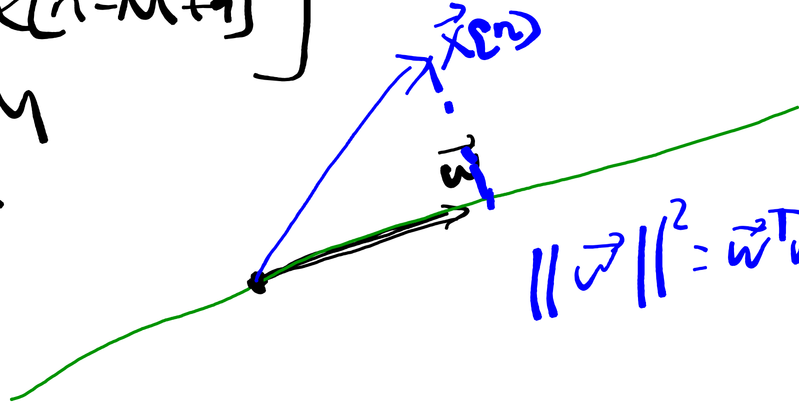

x[n]
z^{-1}
z^{-1}
z^{-1}
w_0
w_1
w_2
w_{M-1}
Σ
y[n]
+
-
ε[n]
d[n]

$$y[n] = \sum_{k=0}^{M-1} w_k \, x[n-k]$$

$$\vec{w} = \begin{bmatrix} w_0 \\ \vdots \\ w_{M-1} \end{bmatrix} \qquad \vec{x}[n] = \begin{bmatrix} x[n] \\ x[n-1] \\ \vdots \\ x[n-M+1] \end{bmatrix}$$

$$y[n] = \vec{w}^T \vec{x}[n]$$

$$\mathbb{R}^M$$



$$\|\vec{w}\|^2 = \vec{w}^T \vec{w}$$

$$\min_{\vec{w}\in\mathbb{R}^M} J(\vec{w}) = E\left[\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)^2\right]$$

$$\frac{\partial J}{\partial w_i} = 0 = \frac{\partial}{\partial w_i} E\left[\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)^2\right]$$

$$= E\left[\frac{\partial}{\partial w_i}\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)^2\right]$$

$$= E\left[2\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right) x[n-i]\right] = 0$$

$$\boxed{E\left[d[n]\,x[n-i]\right] = \sum_{k=0}^{M-1} E\left[x[n-k]\,x[n-i]\right] w_k}$$

$i = 1, \ldots, M-1$

M equations and M unknowns

$$\vec{P} = \begin{bmatrix} E[d[n]\,x[n]] \\ \vdots \\ E[d[n]\,x[n-M+1]] \end{bmatrix} \qquad R = \begin{bmatrix} E[x[n]\,x[n]] & \cdots & E[x[n]\,x[n-M+1]] \\ \vdots & & \\ E[x[n-M+1]\,x[n]] & \cdots & E[x[n-M+1]^2] \end{bmatrix}$$

R: autocorrelation matrix

$\vec{P}$: cross-correlation vector

$$R\vec{w} = \vec{P} \qquad \vec{w}_{opt} = R^{-1}\vec{P}$$

Normal equation

Vector space Interpretation

$\mathcal{X}$ is a set Related to a field $\mathbb{R}$

for $x, y \in \mathcal{X}$ and $\alpha \in \mathbb{R}$

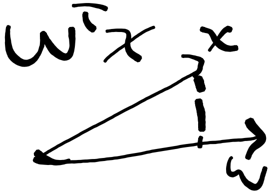
- $z = (x+y) \in \mathcal{X}$
- $\alpha x \in \mathcal{X}$
- $0 \in \mathcal{X}$: $0 + x = x$
- $-x \in \mathcal{X}$: $(-x) + x = 0$

$\mathcal{X}$:= Space of random variables

$$E[X^2] < \infty$$

$$X, Y \in \mathcal{X}$$

$$E[XY] = \langle X, Y \rangle \qquad E[XX] = \langle X, X \rangle = \|X\|^2$$



$$E\big[d[n]\,x[n-i]\big] = E[D X_i]$$

$$P_i = \langle D, X_i \rangle$$

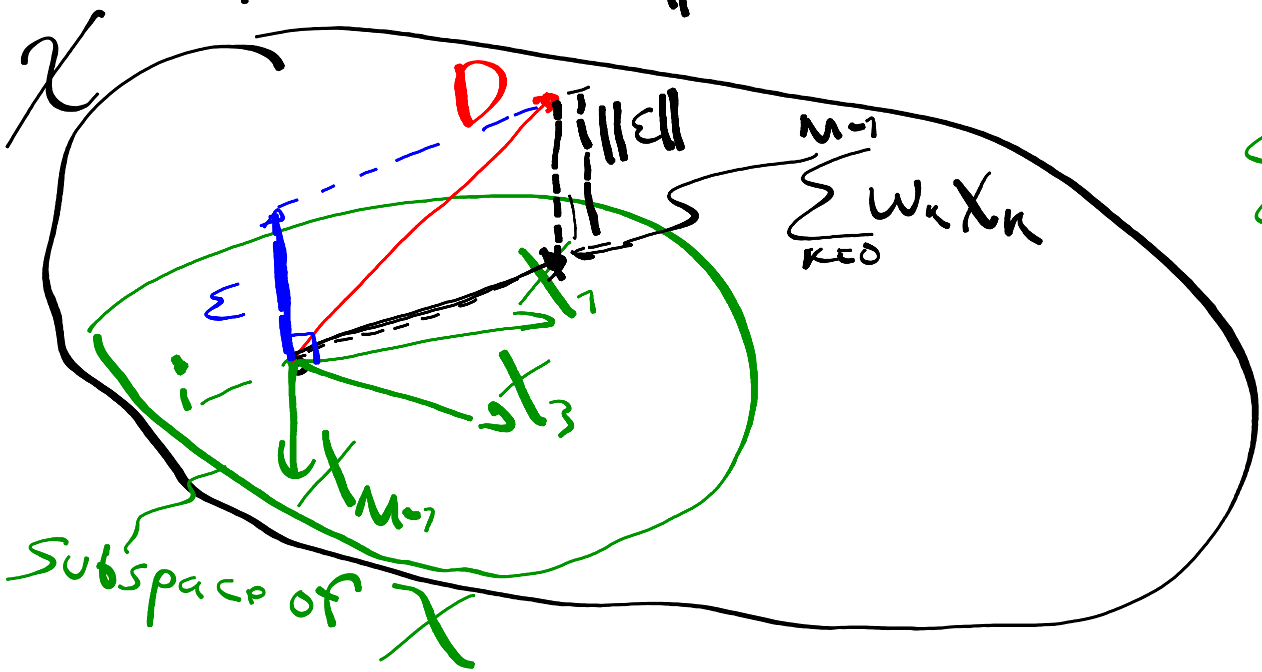
$$R_{ik} = R[i-k] = E\big[x[n-i]\,x[n-k]\big] = E[X_i X_k] = \langle X_i, X_k \rangle$$

$$Y = \sum_{k=0}^{M-1} w_k X_k \qquad \varepsilon = D - Y \Rightarrow J = \|\varepsilon\|^2$$

$$E[\varepsilon^2] = E\left[\left(D - \sum_{k=0}^{M-1} w_k X_k\right)^2\right]$$

$$= \left\langle D - \sum_{k=0}^{M-1} w_k X_k \,,\; D - \sum_{i=0}^{M-1} w_i X_i \right\rangle$$

$$= \left\| D - \sum_{k=0}^{M-1} w_i X_i \right\|^2 = J$$



$$\sum_{i=0}^{M-1} \alpha_i X_i$$

$$0 = \left\langle \varepsilon, \sum \alpha_i X_i \right\rangle$$

$$0 = \langle \varepsilon, X_i \rangle$$

$$\varepsilon = D - y$$

$$= D - \sum_{k=0}^{M-1} w_k X_k$$

$$E[\varepsilon X_i] = E\left[D X_i - \sum_{k=0}^{M-1} w_k X_k X_i\right]$$

$$= E[D X_i] - \sum_{k=0}^{M-1} w_k E[X_k X_i]$$

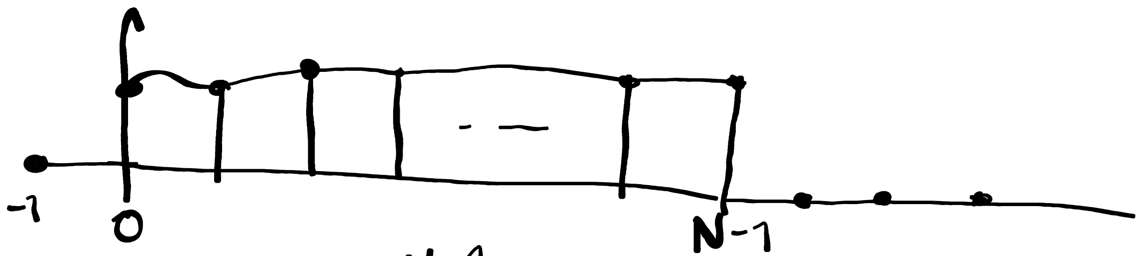
$$P_i - \sum_{k=0}^{M-1} w_k R_{ik} = P_i - P_i = 0$$

Temporal Averages

- Stationarity
- Ergodicity

$x[n] = \{x[0], x[1], \ldots x[N-1]\}$

portion of the signal of length N.



$$J_N = \frac{1}{N}\sum_{m=0}^{N-1}\left(d[m]-y[m]\right)^2$$

$$J_N(\vec{w}) = \frac{1}{N}\sum_{m=0}^{N-1}\left(d[m] - \sum_{k=0}^{M-1} w_k x[m-k]\right)^2$$

$$0 = \frac{\partial J_N}{\partial w_i} = 2\frac{1}{N}\sum_{m=0}^{N-1}\left[\left(d[m] - \sum_{k=0}^{M-1} w_k x[m-k]\right) x[m-i]\right]$$

$$\frac{1}{N}\sum_{m=0}^{N-1} d[m]x[m-i] = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{M-1} w_k x[m-k]x[m-i]$$

$$P_N[i] = \frac{1}{N}\sum_{m=0}^{N-1} d[m]x[m-i] \qquad R_N[i-k] = \frac{1}{N}\sum_{m=0}^{N-1} x[m-k]x[m-i]$$

$$P_N[i] = \sum_{k=0}^{M-1} w_k R_N[i-k] \quad \text{for } i=0,\ldots,M-1$$

normal equation (Based on Samples and Time averages)

Numerical Solutions

$$\vec{w}_{opt} = R^{-1}\vec{P}$$

$$= R_N^{-1}\vec{P}_N$$

is R invertible?

$$\vec{w}^{(t+1)} = (I - \mu R)\vec{w}^{(t)} + \mu\vec{P}$$

$$\vec{w}^{(0)} = \vec{0}$$

$$\|\vec{v}\| > \|(I - \mu R)v\| \qquad \mu > 0$$

$$w^{(t+1)} = \mu\left[\sum_{m=0}^{t}(I - \mu R)^m\right]\vec{P}$$

$t \to \infty$

$$R^{-1}\vec{P}$$

R is singular
the iterative algorithm
finds a solution (not unique)

R is close to singular (condition number large)

$R\vec{w}' = \vec{p} + \epsilon$ $\quad \epsilon$ is small

$R\vec{w} = \vec{p}$ $\quad \|\vec{w}' - \vec{w}\|$ may be extremely large

iterative Solution is more robust

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Wiener filter (Luis)