# Notes on Wiener Filters

Luis Gonzalo Sánchez Giraldo
Computational Neuro-Engineering Laboratory
University of Florida

Spring 2012

## 1 Basic Setting

Here, we will only address the discrete time formulation of the Wiener filter. Consider a widesense stationary random process $x[n]$ in the presence of additive noise $n[n]$ independent of $x[n]$. so that we have $u[n] = x[n] + n[n]$ sent through a time invariant linear system whose output is denoted by $y[n]$. The signal $y[n]$ will approximate the output of a linear operator $L$ on $x[n]$ by minimizing the mean square error:

$$J = \mathrm{E}\left\{\varepsilon^2[n]\right\} = \mathrm{E}\left\{(L(x[n]) - y[n])^2\right\} \tag{1}$$

The relation between $X[n]$ and $Y[n]$ it is obtained through a convolution operation, which in the general case is expressed as,

$$y[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] \tag{2}$$

The goal is to find the system $h[n]$ such that the cost $J$ in (1) is minimized. Based on (2) we can enumerate three main problem related to the above setup:

- The pure prediction problem, where $h[n] = 0$ for $n < n_0$, for $n_0 \geq 0$ and $n[n] = 0$. In this case the linear system $h[n]$ is said to be causal and the effect of additive noise is neglected by the mode. Equation (2) reduces to,
$$y[n] = \sum_{k=n_0}^{\infty} h[k]x[n-k] \tag{3}$$
- The prediction and estimation problem, where $h[n] = 0$ for $n < 0$ and $n[n] \neq 0$.
- The estimation , where $h[n]$ can be different from zero for $n < 0$

In this notes we will restrict to first case (prediction problem) and assume a finite impulse response (FIR) structure for $h[n]$. These considerations yield the following expansion of the output of the filter:

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k] = \sum_{k=0}^{M-1} w_k x[n-k] \tag{4}$$

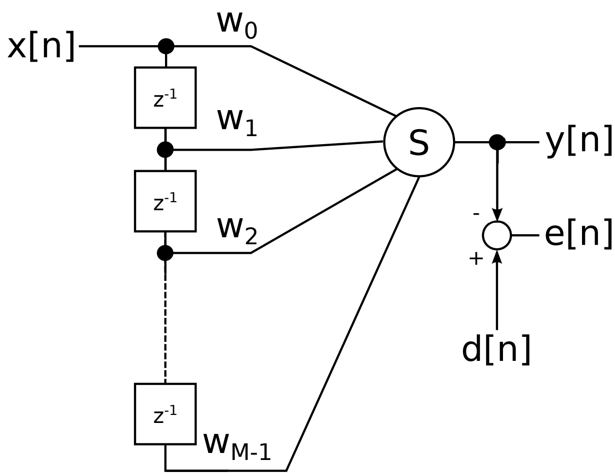

Figure 1: Block diagram of the structure of causal FIR filter considered in the prediction problem

It is important to highlight the notational change made in the rightmost hand side term in (4) which will be useful in understanding the filter. The values of the impulse response $h[n]$ are written as entries of a $M$-dimensional vector $\mathbf{w} = (w_0,\ w_1, \ldots,\ w_{M-1})^{\mathrm{T}}$, which we will assume to have real entries, in other words, $\mathbf{w} \in \mathbb{R}^M$. A block structure of the filter in (4) is depicted in Figure 1.

## 2 Solution to the Optimal Filter Problem

Based on the delay tap structure in Figure 1, let us write the current value of the signal $x[n]$ and the $M-1$ previous values in the form of a vector

$$\mathbf{x}[n] = (x[n]; x[n-1], x[n-2], \ldots,\ x[n-M+1])^{\mathrm{T}},$$

such that (4) can be written using matrix notation as:

$$y[n] = \sum_{k=0}^{M-1} w_k x[n-k] = \mathbf{w}^{\mathrm{T}}\mathbf{x}[n] \tag{5}$$

The above change is important to stress the relation between the convolution operation with an FIR filter at time $n$ and the inner product operation in the Euclidean space $\mathbb{R}^M$; basically we represent an $M$-sample window of $x[n]$ as a vector in $\mathbb{R}^M$ and the filter action as a scaled projection of this vector $\mathbf{x}[n]$ onto the line spanned by the vector $\mathbf{w}$ (see Figure 2). Acquaintance with the matrix notation is also very important due to its widespread use in the adaptive filter literature. In general, we will assume the outputs of the operator $L$ described in the previous section are given in the form a signal $d[n]$ called *desired* signal. The objective in (1) in terms of $d[n]$ is written as:

$$J = \mathrm{E}\left\{\boldsymbol{\varepsilon}^2[n]\right\} = \mathrm{E}\left\{(d[n]-y[n])^2\right\}. \tag{6}$$

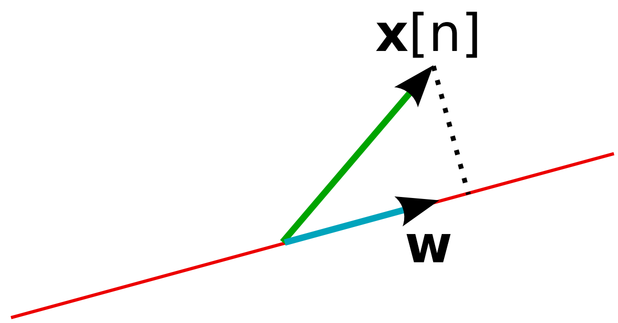


Figure 2: Inner product operation related to the application of an FIR filter at time $n$

The optimal filter is obtained as a solution of the following optimization problem:

$$\min_{\mathbf{w}\in\mathbb{R}^M} J(\mathbf{w}) = \mathrm{E}\left\{\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)^2\right\} \tag{7}$$

The solution can be obtained from the first order necessary conditions, that is, by equating the vector of partial derivatives $\frac{\partial J}{\partial w_i}$ to zero. The partial derivatives of the objective $J$ are:

$$\begin{aligned}
\frac{\partial J}{\partial w_i} &= \frac{\partial}{\partial w_i}\mathrm{E}\left\{\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)^2\right\} \qquad (8)\\
&= \mathrm{E}\left\{\frac{\partial}{\partial w_i}\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)^2\right\}\\
&= \mathrm{E}\left\{2\left(-\frac{\partial}{\partial w_i}\sum_{k=0}^{M-1} w_k x[n-k]\right)\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)\right\}\\
&= 2\,\mathrm{E}\left\{(-x[n-i])\left(d[n] - \sum_{k=0}^{M-1} w_k x[n-k]\right)\right\}\\
&= -2\,\mathrm{E}\{d[n]x[n-i]\} + 2\sum_{k=0}^{M-1} w_k\,\mathrm{E}\{x[n-k]x[n-i]\}
\end{aligned}$$

And the optimal filter is the set of weights $\{w_i\}$ satisfying $M$ equations:

$$\mathrm{E}\{d[n]x[n-i]\} = \sum_{k=0}^{M-1} w_k\,\mathrm{E}\{x[n-k]x[n-i]\} \quad \text{for } i = 0, 1, \ldots, M-1. \tag{9}$$

Lets call $P[i] = P_i = \mathrm{E}\{d[n]x[n-i]\}$ the cross-correlation function between $d[n]$ and $x[n-i]$ and $R[i-k] = R_{ik} = \mathrm{E}\{x[n-k]x[n-i]\}$ the autocorrelation function of $x[n]$. Arranging the values of these functions in matrix form as: $\mathbf{P} = (P_0, P_1, \ldots, P_{M-1})^{\mathrm{T}}$, and

$$\mathbf{R} = \begin{bmatrix} R[0] & R[1] & \cdots & R[M-1] \\ R[1] & R[0] & \cdots & R[M-2] \\ \vdots & \vdots & \ddots & \vdots \\ R[M-1] & R[M-2] & \cdots & R[0] \end{bmatrix};$$

yields the matrix equation,

$$\mathbf{Rw} = \mathbf{P}, \tag{10}$$

commonly know as Wiener-Hopf equations or Normal equations. The optimal filter is the vector $\mathbf{w}$ for which (10) holds is obtained from the inverse of $\mathbf{R}$ as

$$\mathbf{w}_{\mathrm{opt}} = \mathbf{R}^{-1}\mathbf{P}. \tag{11}$$

## 3 Vector Space Interpretation

Consider a linear vector space $\mathcal{X}$ were the elements are random variables $X$ such that $\mathrm{E}\left\{X^2\right\} < \infty$. In other words, each random variable can be seen as a vector in this space. The inner product between $X$ and $Y$ in $\mathcal{X}$ corresponds to $\mathrm{E}\{XY\} = \langle X, Y\rangle$. For instance, the cross-correlation between $d[n]$ and $x[n-i]$ can be understood as the inner product between two random variables $D$ and $X_i$, that is, $P_i = \langle D, X_i\rangle$. Since $x[n]$ is a widesense stationary process our averages $\mathrm{E}\{d[n]x[n-i]\}$ only depend on the time difference $i$; similarly, $R_{ik} = \mathrm{E}\{x[n-k]x[n-i]\}$ only depends on the difference $i-k$, and thus $R_{ik} = \langle X_i, X_k\rangle$. The output $y[n]$ of the linear system at each time $n$ can be seen as a random variable $Y$ that correspond to a linear combinations of $M$ random variables $X_i \in \mathcal{X}$. The mean squared error can be written as

$$\mathrm{E}\left\{\left(D - \sum_{i=0}^{M-1} w_i X_i\right)^2\right\} = \left\langle D - \sum_{i=0}^{M-1} w_i X_i, D - \sum_{i=0}^{M-1} w_i X_i\right\rangle = \left\|D - \sum_{i=0}^{M-1} w_i X_i\right\|^2 \tag{12}$$

The space of all vectors of the form $\sum_{i=0}^{M-1} \alpha_i X_i$ for all $\alpha_i \in \mathbb{R}$, forms a subspace of $\mathcal{X}$. The solution to the optimal filter is set of weights $\mathbf{w}_{\mathrm{opt}}$ for which the squared distance between $D$ and the vector $\sum_{i=0}^{M-1} w_i X_i$ is minimized. This corresponds to the orthogonal projection of $D$ onto the subspace spanned by the set of vectors $\{X_i\}$ for $i = 0, 1, \ldots, M-1$. Figure 3 shows the geometrical interpretation of the solution to the optimal filtering problem. In general the difference between desired signal $d[n]$ and the output $y[n]$ of the optimal filter cannot be made zero. Thus we have the error signal $\varepsilon[n]$ for which we have that $\mathrm{E}\{X_i\varepsilon\} = 0$ for all $i$ and therefore $\mathrm{E}\{Y\varepsilon\} = 0$. The error vector $\varepsilon \in \mathcal{X}$ is therefore orthogonal to the subspace spanned by the $\{X_i\}$'s. A simple proof of the above can be obtained as

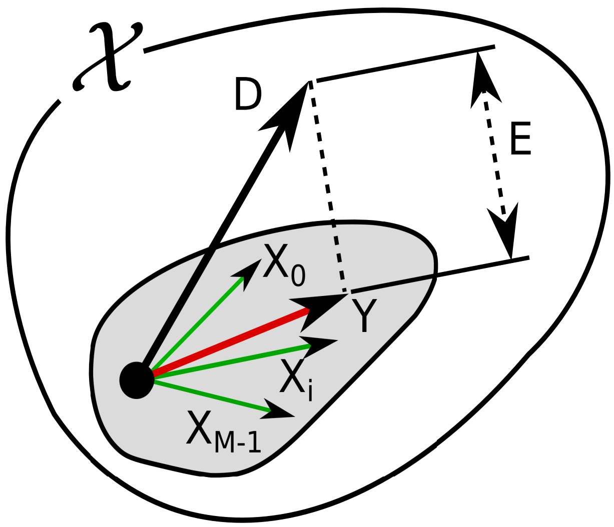


Figure 3: Vector space interpretation of the solution to the optimal filter problem

follows:

$$
\begin{aligned}
\varepsilon &= D - Y && (13)\\
&= D - \sum_{k=0}^{M-1} w_k X_k && (14)\\
\varepsilon X_i &= DX_i - \sum_{k=0}^{M-1} w_k X_k X_i && (15)\\
\mathrm{E}\{\varepsilon X_i\} &= \mathrm{E}\{DX_i - \sum_{k=0}^{M-1} w_k X_k X_i\} = P_i - \sum_{k=0}^{M-1} R_{ik} w_k = 0 && (16)
\end{aligned}
$$

for all $i = 0, 1, \ldots, M-1$.

## 4 Temporal Averages

In practice, Wiener filters are obtained using the temporal averages to approximate the cross-correlation and autocorrelation functions involved the solution (Equation (10)). Temporal averages assume ergodicity of the stationary random process. Consider a segment of $N$ points of a realization of a stationary stochastic process that is also ergodic. Let us define the error as,

$$
\begin{aligned}
J_N = \frac{1}{N}\sum_{m=0}^{N-1} e^2[m] = \frac{1}{N}\sum_{m=0}^{N-1} (d[m] - y[m])^2 \\
= \frac{1}{N}\sum_{m=0}^{N-1}\left(d[m] - \sum_{k=0}^{M-1} w_k x[m-k]\right)^2,
\end{aligned} \quad (17)
$$

and find the minimizer $\mathbf{w}_{\text{opt}}$ by computing $\frac{\partial J_N}{\partial w_i}$ and equating to zero. We obtain:

$$\frac{\partial J_N}{\partial w_i} = -\frac{1}{N}\sum_{m=0}^{N-1}\left[\left(d[m] - \sum_{k=0}^{M-1} w_k x[m-k]\right) x[m-i]\right] = 0, \tag{18}$$

for $i = 0, 1, \ldots, M-1$. This gives a set of equations:

$$\frac{1}{N}\sum_{m=0}^{N-1} d[m]x[m-i] = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{M-1} w_k x[m-k]x[m-i] \tag{19}$$

Defining $P_N[i] = P_i = \frac{1}{N}\sum_{m=0}^{N-1} d[m]x[m-i]$ and $R_N[i-k] = R_{ik} = \frac{1}{N}\sum_{m=0}^{N-1} x[m-k]x[m-i]$, yields:

$$P_N[i] = \sum_{k=0}^{M-1} w_k R_N[i-k], \tag{20}$$

this correspond to the normal equations we introduced before but the cross-correlation and autocorrelation have been replaced by time averages with a finite number of points.

# 5 Numerical Solution of the Optimal Vector

If the inverse of $\mathbf{R}$ exist, we can compute the solution directly using (11). However, computing the inverse without any considerations about the structure of matrix leads to $\mathcal{O}(M^3)$ calculations where $M \times M$ is the size of $\mathbf{R}$. For Toeplitz matrices the complexity can be dropped to $\mathcal{O}(M^2)$. Another approach is the iterative solution such as:

$$\mathbf{w}^{(t+1)} = (\mathbf{I} - \mu\mathbf{R})\,\mathbf{w}^{(t)} + \mu\mathbf{P} \tag{21}$$

with initial condition $\mathbf{w}^{(0)} = \mathbf{0}$ and $\mu > 0$ such that

$$\|\mathbf{v}\| > \|\,(\mathbf{I} - \mu\mathbf{R})\,\mathbf{v}\|$$

this method requires $\mathcal{O}(M^2)$ operations per iteration and solution should converge to $\mathbf{R}^{-1}\mathbf{P}$. Let us rewrite (21) as follows:

$$\mathbf{w}^{(t+1)} = \mu\left[\sum_{m=0}^{t} (\mathbf{I} - \mu\mathbf{R})^m\right]\mathbf{P}$$

taking the limit $t \to \infty$ we have $\mu\left[\sum_{m=0}^{t} (\mathbf{I} - \mu\mathbf{R})^m\right]\mathbf{P} \to \mathbf{R}^{-1}\mathbf{P}$. Direct method may be more efficient,in general. Nevertheless, there are cases were iterative method is advantageous, for instance:

- When $\mathbf{R}$ is singular, iterative method can find a solution, albeit not unique.
- When $\mathbf{R}$ is close to singular (large condition number), the iterative solution tends to be more robust (small perturbations of the signal won't affect the optimal solution).