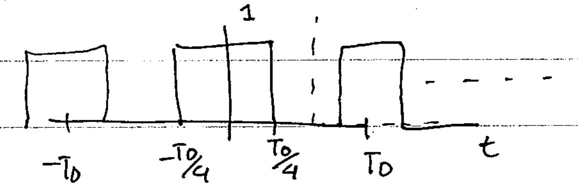
SEPTEMBER 16 LECTURE.

PROOF OF SHIFTING PROPERTY

1. SPECTRUM OF



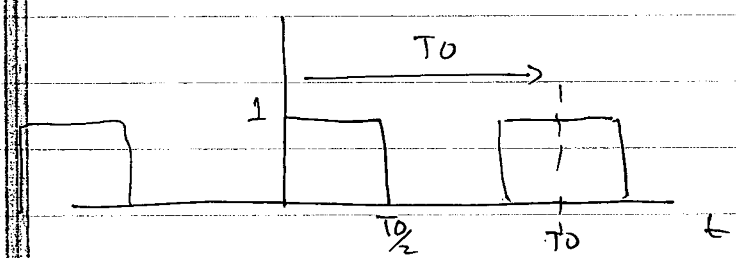
$$x(t) = \begin{cases} 1 & -\frac{T_0}{4} \le t \le T_0/4 \\ 0 & \text{OTHERWISE} \end{cases} \qquad |t| < \frac{T_0}{2}$$

$$a_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-j\frac{2\pi k t}{T_0}}\,dt = \frac{-1}{\cancel{T_0}}\,\frac{\cancel{T_0}}{j2\pi k}\left[e^{-j\frac{\cancel{2}\pi k}{2\cancel{4}}} - e^{j\frac{\cancel{2}\pi k}{\cancel{4}2}}\right]$$

$$= \frac{1}{\pi k}\sin\frac{\pi}{2}k \qquad k = \dots -1, 0, 1, \dots \infty$$

$$a_0 = \frac{1}{2}$$

SPECTRUM OF (BOOK pp53)



$$y(t) = \begin{cases} 1 & 0 < t \le T_0/2 \\ 0 & |t| < \frac{t_0}{2} \end{cases}$$

NOTICE THAT $y(t) = x(t - \frac{T_0}{4})$

CHANGE OF VARIABLES $t - \frac{T_0}{4} = u \Rightarrow dt = du$

NOTE: IN GENERAL ALSO NEED TO WORRY ABOUT LIMITS OF INTEGRATION, BUT HERE IT IS ALWAYS OVER A PERIOD.

So SPECTRUM OF $y(t)$ IS.

$$b_k = \frac{1}{T_0}\int_0^{T_0} y(u)\, e^{-j\frac{2\pi k\left(u+\frac{T_0}{4}\right)}{T_0}}\, du$$

$$= e^{-j\frac{\pi k}{2}}\, \frac{1}{T_0}\int_0^{T_0} y(u)\, e^{-j\frac{2\pi k u}{T_0}}\, du = e^{-j\frac{\pi k}{2}}\, a_k$$

$$= e^{-j\frac{\pi k}{2}}\left(\frac{1}{\pi k}\,\sin\frac{\pi k}{2}\right) = \frac{e^{-j\frac{\pi k}{2}}}{\pi k}\left(\frac{e^{j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}}}{2j}\right)$$

$$= \frac{1}{j2\pi k}\left(1 - e^{-j\pi k}\right) \qquad \text{SAME AS IN BOOK}$$

—— // ——

SPECTRUM OF TRIANGULAR WAVE

$$x(t) = \begin{cases} 2t/T_0 & 0 < t < \frac{1}{2}T_0 \\ \frac{2(T_0-t)}{T_0} & \frac{1}{2}T_0 < t < T_0 \end{cases}$$

$$a_0 = \frac{1}{2}$$

$$a_k = \underbrace{\frac{1}{T_0}\int_0^{T_0/2} \frac{2t}{T_0}\, e^{-j\frac{2\pi k t}{T_0}}\, dt}_{(1)} + \frac{1}{T_0}\underbrace{\int_{T_0/2}^{T_0} \frac{2(T_0-t)}{T_0}\, e^{-j\frac{2\pi k t}{T_0}}\, dt}_{(2)}$$

WORK : (1) (OTHER IS SIMILAR)

$$\frac{2}{T_0^2}\int_0^{T_0/2} \underbrace{t}_{u}\; \underbrace{e^{-j\frac{2\pi k t}{T_0}}}_{v'}\, dt \qquad \left(\int u v' = uv\Big| - \int u' v\right)$$

$$= \frac{2}{T_0^2}\left[\underbrace{\frac{t\, e^{-j\frac{2\pi k t}{T_0}}}{-j\frac{2\pi k}{T_0}}\Bigg|_0^{T_0/2}}_{A} + \underbrace{\frac{1}{j\frac{2\pi k}{T_0}}\int_0^{\frac{T_0}{2}} e^{-j\frac{2\pi k t}{T_0}}\, dt}_{B}\right]$$

(A) $\frac{2}{T_0^2}\left[\frac{T_0^2}{-j4\pi k} e^{-j\pi k} - 0\right] = -\frac{e^{-j\pi k}}{j2\pi k}$

(B) $\frac{2}{T_0^2}\left[\frac{-1}{\left(j\frac{2\pi k}{T_0}\right)^2}\left(e^{-j\pi k} - 1\right)\right] = \frac{2}{T_0^2}\left[\frac{T_0^2}{4\pi^2 k^2}\left(e^{-j\pi k} - 1\right)\right]$

$= \frac{\left(e^{-j\pi k} - 1\right)}{2\pi^2 k^2}$

So (1) is $(a_k = a_{k1} + a_{k2})$

$$a_{k1} = \frac{-e^{-j\pi k}}{j2\pi k} + \frac{e^{-j\pi k} - 1}{2\pi^2 k^2}$$

Now (2)

$$a_{k2} = \underbrace{\frac{1}{T_0}\int_{T_0/2}^{T_0} 2e^{-j\frac{2\pi k t}{T_0}}\,dt}_{C} - \underbrace{\frac{1}{T_0^2}\int_{T_0/2}^{T_0} 2t\,e^{-j\frac{2\pi k t}{T_0}}\,dt}_{D}$$

$$C = -\frac{1}{j\pi k}\left[1 - e^{-j\pi k}\right]$$

$$D = \frac{1}{j\pi k}\left[1 - \frac{1}{2}e^{-j\pi k}\right] - \frac{1}{2\pi^2 k^2}\left(1 - e^{-j\pi k}\right)$$

So FINALLY!

$$a_k = \frac{1}{j\pi k}\left(-\frac{e^{-j\pi k}}{2} \cancelto{0}{- 1 + e^{-j\pi k} + 1 - \frac{1}{2}e^{-j\pi k}}\right) + \frac{1}{2\pi^2 k^2}\left(e^{-j\pi k} - 1 - 1 + e^{-j\pi k}\right)$$

$$= \frac{1}{\pi^2 k^2}\left(-1 + e^{-j\pi k}\right)$$

(SEE BOOK pp 56)