Problem 1 (15 points)

Given the signal

$$x(t) = 0.5\cos(2\pi 10t + \pi/2) + \mathrm{Im}\{e^{j(2\pi 10t+\pi/3)}\}$$

Write x(t) in the form of $A\sin(wt + \phi)$.

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$$\mathrm{Im}\left\{e^{j(2\pi 10t+\frac{\pi}{3})}\right\} = \sin\left(2\pi 10t + \frac{\pi}{3}\right)$$

Also $\cos\left(x+\frac{\pi}{2}\right) = -\sin x$ So

$$x(t) = -0.5\sin(2\pi 10t) + \sin\left(2\pi 10t+\frac{\pi}{3}\right)$$

FROM PHASOR ADDITION $\sum A_k \cos(\omega_0 t + \phi_k) = A\cos(\omega_0 t + \phi)$ with

$$A e^{j\phi} = \sum_k A_k e^{j\phi_k}$$

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So $A_1 = -0.5$ $\phi_1 = 0$ $A e^{j\phi} = -0.5 + e^{j\frac{\pi}{3}} =$

$A_2 = 1$ $\phi_2 = \frac{\pi}{3}$

$$= -0.5 + \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = \cancel{-0.5} + \cancel{0.5} + j\frac{\sqrt{3}}{2}$$

therefore $A = \frac{\sqrt{3}}{2}$ AND $\phi = 90^\circ = \frac{\pi}{2}$

AND $x(t) = \frac{\sqrt{3}}{2}\sin\left(2\pi 10t + \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}\cos(2\pi 10t)$

Problem 2 (15 points)

Consider the signal

$$x(t) = \cos(2\pi 10t + \pi/5) + 3\cos(2\pi 45t)\sin(2\pi 10t)$$

2a) Compute the fundamental frequency in Hertz.

2b) Calculate and plot the two sided spectrum of x(t).

$$\cos(2\pi 45t)\sin(2\pi 10t) = \left(\frac{e^{j2\pi 45t} + e^{-j2\pi 45t}}{2}\right)\left(\frac{e^{j2\pi 10t} - e^{-j2\pi 10t}}{2j}\right) =$$

$$= \frac{1}{4j}\left(e^{j2\pi 55t} + e^{-j2\pi 35t} - e^{+j2\pi 35t} - e^{-j2\pi 55t}\right)$$

2a) 3 frequencies: 10 Hz, 35 Hz, 55 Hz

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Periods $\frac{1}{10}$, $\frac{1}{35}$, $\frac{1}{55}$ G.C.D $= \frac{1}{5}$ so

(2×5) (7×5) (11×5) $f_0 = 5\,Hz$

2b)

$$X(\omega) = \frac{e^{j\frac{\pi}{5}}}{2}\left(e^{j2\pi 10} + e^{-j2\pi 10}\right) + \frac{3}{4j}\left[e^{j2\pi 55} + e^{-j2\pi 35} - e^{+j2\pi 35} - e^{-j2\pi 55}\right]$$

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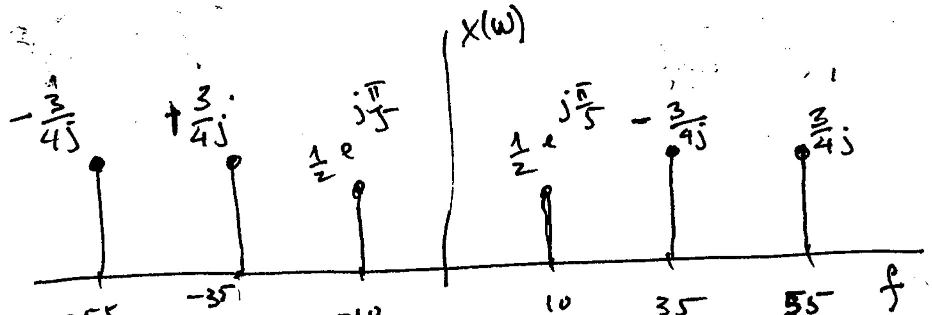

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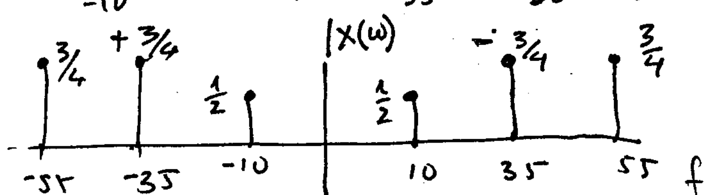
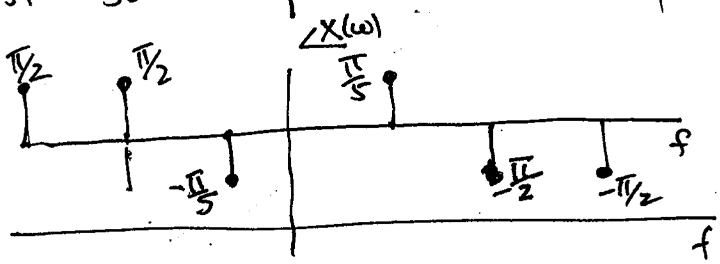


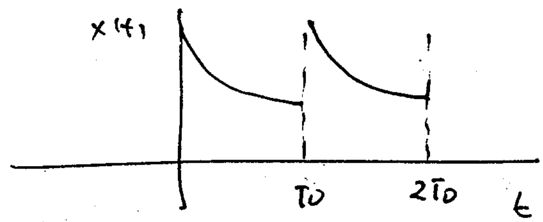


Problem 3 (15 points)

Calculate the Fourier series decomposition of the periodic signal (with period T_0)

$$x(t) = e^{-2\pi t/T_0} \qquad 0 \le t \le T_0$$

Plot the magnitude and phase spectrum (only for k=0, $k = \pm 1$). Show all your work.



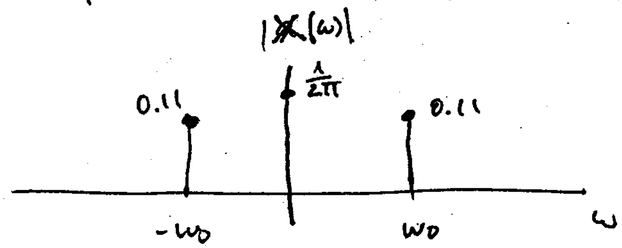
$$a_k = \frac{1}{T_0}\int_0^{T_0} e^{-\frac{2\pi t}{T_0}} e^{-j\frac{2\pi k t}{T_0}} dt = \frac{1}{T_0}\int_0^{T_0} e^{-\frac{2\pi t}{T_0}(1+jk)} dt =$$

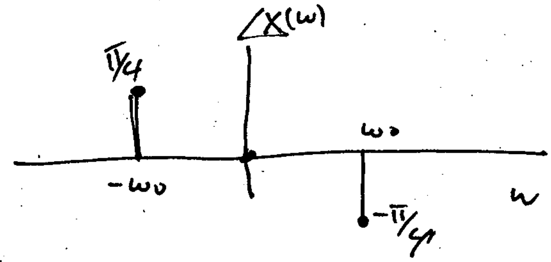
$$= \frac{1}{\cancel{T_0}} \frac{\cancel{T_0}}{-2\pi k(1+jk)} e^{-\frac{2\pi t}{T_0}(1+jk)} \Big|_0^{T_0} = \frac{1}{2\pi(1+jk)}\left(1 - e^{-2\pi(1+jk)}\right)$$

$k=0$ $\quad a_0 = \frac{1}{2\pi}\left(1 - e^{-2\pi}\right) \approx \frac{1}{2\pi}$

$k=1$ $\quad a_1 = \frac{1}{2\pi(1+j)}\left(1 - e^{-2\pi(1+j)}\right) = \frac{1}{2\pi(\sqrt{2}e^{j\frac{\pi}{4}})}\left(1 - e^{-2\pi}\underbrace{\cancel{e^{-j2\pi}}}_{=1}\right) \approx 0.11\, e^{-j\frac{\pi}{4}}$

$k=-1$ $\quad a_{-1} = 0.11\, e^{j\frac{\pi}{4}}$ since $x(t)$ is real signal.





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Problem 4 (20 points)

A periodic signal x(t) with ~~period~~ fundamental frequency 50 Hz has the following magnitude spectrum

$$|a_k| = \begin{cases} 1 & k=0 \\ \dfrac{1}{2|k|} & k=\pm 1, \pm 2 \\ 0 & otherwise \end{cases}$$

Now you still do not know the phase spectrum. Knowing that x(t) is a real signal, choose a non-zero phase spectrum and write a simplified equation for x(t). Justify your choice.

5 $x(t)$ real $\Longrightarrow$ Phase is antisymmetric

I $\longrightarrow$ SELECT THESE VALUES (you could select others)

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$\angle a_1 = \frac{\pi}{4} \Longrightarrow \angle a_{-1} = -\frac{\pi}{4}$

$\angle a_2 = \pi/2 \Longrightarrow \angle a_{-2} = -\frac{\pi}{2}$

0 otherwise.

So $$x(t) = \sum_{k=-2}^{2} a_k e^{j\frac{2\pi k t}{0.02}} =$$

10 $$= \frac{1}{4} e^{-j\frac{\pi}{2}} e^{-j\frac{4\pi t}{0.02}} + \frac{1}{2} e^{-j\frac{\pi}{4}} e^{-j\frac{2\pi t}{0.02}} + 1 + \frac{1}{2} e^{j\frac{\pi}{4}} e^{j\frac{2\pi t}{0.02}} + \frac{1}{4} e^{j\frac{\pi}{2}} e^{j\frac{4\pi t}{0.02}}$$

$$= 1 + \frac{1}{2}\cos\left(\frac{4\pi t}{0.02} + \frac{\pi}{2}\right) + \cos\left(\frac{2\pi t}{0.02} + \frac{\pi}{4}\right)$$

Problem 5 (20 points)

Given the signal

x(t)= cos(0.1πt)[u(t+5)-u(t-5)]

5a) Plot x(t) in the time domain
5b) Evaluate its Fourier transform
5c) Plot the signal y(t)=cos(0.1πt)[u(-t-5)+u(t-5)] in the time domain
5d) Determine the Fourier transform of y(t)

5a) $\qquad 0.1\pi t = \frac{\pi}{2} \Rightarrow t=5$

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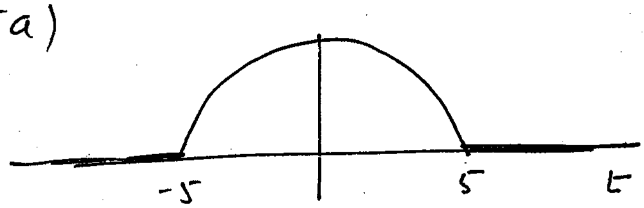

5b) $X(j\omega) = \int_{-5}^{5} \cos(0.1\pi t)\, e^{-j\omega t}\, dt = \underbrace{\frac{1}{2}\int_{-5}^{5} e^{j0.1\pi t} e^{-j\omega t}\, dt}_{A} + \underbrace{\frac{1}{2}\int_{-5}^{5} e^{-j0.1\pi t} e^{-j\omega t}\, dt}_{B}$

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$A = \frac{1}{2}\int_{-5}^{5} e^{jt(0.1\pi-\omega)}\, dt = \left.\frac{e^{jt(0.1\pi-\omega)}}{2(j(0.1\pi-\omega)}\right|_{-5}^{5} = \frac{\sin 5(0.1\pi-\omega)}{0.1\pi-\omega}$

$B = -\frac{\sin(5(0.1\pi+\omega))}{0.1\pi+\omega} \Rightarrow X(j\omega) = \frac{\sin(5(0.1\pi-\omega))}{0.1\pi-\omega} - \frac{\sin(5(0.1\pi+\omega))}{0.1\pi+\omega}$

5c)

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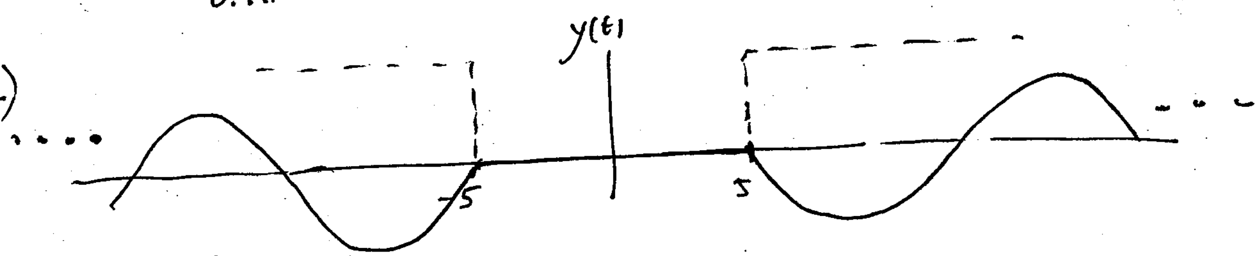

5d) Notice that $\cos(0.1\pi t) = x(t) + y(t)$

Since $\cos(0.1\pi t) \xleftrightarrow{\mathcal{F}} \pi\,\delta(\omega-0.1\pi) + \pi\,\delta(\omega+0.1\pi)$

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then

$Y(j\omega) = \pi\,\delta(\omega-0.1\pi) + \pi\,\delta(\omega+0.1\pi) - X(j\omega)$

Problem 6 (15 points)

Consider the signal $x(t) = 2\sin(2\pi 80t)$.
The signal x(t) is supposed to be sampled according to the Shannon Nyquist theorem. The A/D converter has the following sampling frequencies available: 80, 150, 200Hz.

6a) Choose the appropriate sampling frequency and write an equation for x[n]. Sketch the magnitude spectrum of the discrete time signal, and label appropriately the digital frequency axis.

6b). Write an equation for the discrete time signal x[n] when the sampling frequency is fs =80 Hz. Plot both x[n] and the corresponding spectrum.

FOR 6b
6c) State the frequency of the reconstructed version of x[n], $\hat{x}(t)$, when you use the ideal D/A converter. Justify your answer.

6a) $f_{MAX} = 80\text{Hz} \Longrightarrow f_s = 200\text{ Hz}$ TO MEET Nyquist theorem ($f_s > 2 f_{MAX}$)

$\hat{w} = \frac{2\pi \times 80}{200} = \frac{4}{5}\pi$ SO $x[n] = 2\sin(\frac{4}{5}\pi n)$

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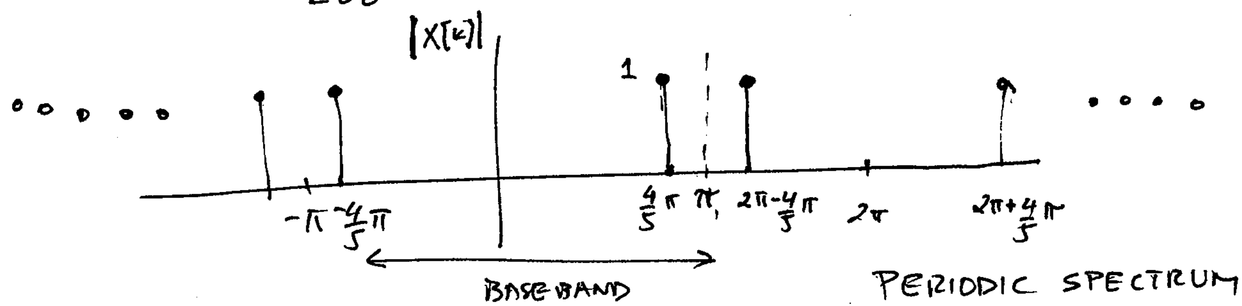

6b) WHEN $f_s = 80\text{Hz}$ $\hat{w} = \frac{2\pi\, 80}{80} = 2\pi$ SO $\hat{x}[n] = 2\sin(2\pi n) \equiv 0$

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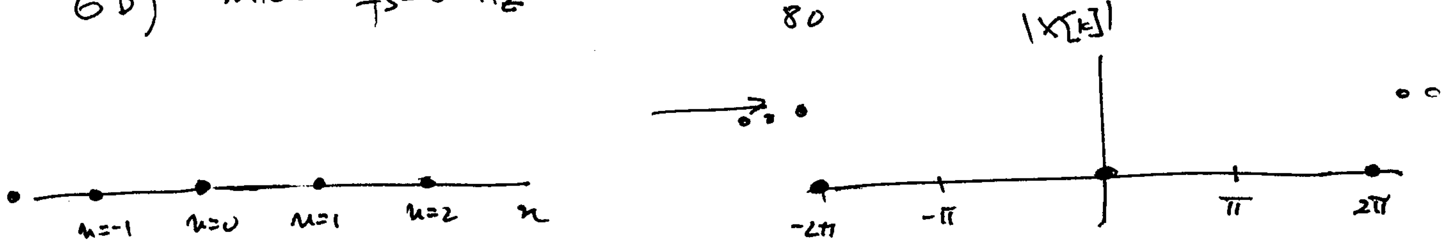

6c) TO RECONSTRUCT 6b IDEALLY THE TIME SIGNAL IS ALWAYS ZERO $\hat{x}(t) = 0$

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