EEL6825 – Fall 99
Exam 1 October 6, 1999

NAME: ______________________________

This exam is open-book and calculator. You may use any books or papers that you like. There are four problems and one extra credit in this exam, you have three full class periods. State your assumptions and reasoning for each problem. Justify your steps and clearly indicate your final answers.

1	**/25**
2	**/25**
3	**/25**
4	**/25**
*****	**/5**
TOTAL	

1. (25 points)

You are given the following two 1-D distributions:

$$p(x|\omega_1) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$p(x|\omega_2) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

Unfortunately the a priori probabilities $P(\omega_1)$ and $P(\omega_2)$ are unknown!!! Answer the following questions.

(a) (5 points)

For what set of a priori probabilities is the Bayes error minimum? What is the value of the Bayes error in this case?

(b) (10 points)

Find the a priori probabilities such that the Bayes classifier classifies a point x as:

$$\begin{array}{ll} \omega_1 & \text{if } x \leq 1/3 \\ \omega_2 & \text{if } x > 1/3 \end{array}$$

(c) (10 points)

Find the a priori probabilities such that the component of the Bayes error due to each class is the same.

2. (25 points)

   You are given two normal distributions with the following means and covariance matrices:

$$\mu_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mu_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Sigma_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

   Alas the a priori probabilities $P(\omega_1)$ and $P(\omega_2)$ are again unknown!!!

   (a) (5 points)

   Sketch one equal probability contour for each distribution. (Make sure that the contour you draw for each distribution specifies the same probability value).

(b) (10 points)

Compute values of $P(\omega_1)$ and $P(\omega_2)$ such that the Bayes discrimination boundary is given by $x_1^2 + x_2^2 = 1$. Show all of your work.

(c) (10 points)

Calculate the Bhatacharrya bound for this problem using the a priori probability values derived in (b).

3. (25 points)

Consider the following probability distribution:

$$p(x) = \begin{cases} (k+1)x^k & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) (5 points)

For what values of k is this distribution valid? Verify that this distribution integrates to one for all valid values of k.

(b) (15 points)

Given N data points $x^{(1)}, x^{(2)} \ldots, x^{(N)}$ sampled from a distribution, derive a formula for the maximum likelihood value of k for the distribution.

(c) (5 points)
Suppose $N = 3$ and $x^{(1)} = e^0 = 1$, $x^{(2)} = e^{-1}$, $x^{(3)} = e^{-2}$. What the numerical value of the maximum likelihood estimate of k?

4. (25 points) Short Answer.

(a) (5 points)
Is it possible for the Bhattacharrya bound to be less than the Bayes error? Assume that you are given the correct distributions, parameters and a priori probabilities. Explain why or why not.

(b) (5 points)
Is it possible for the Bhattacharrya bound to be greater than 1/2 for a two-class classification problem? Explain why or why not.

(c) (5 points)

Given the following data points, find the $\mathbf{w}$ vector that minimizes the Fisher criterion. Assume $P(\omega_1) = P(\omega_2)$. Make sure that you normalize $\mathbf{w}$ and that it points in the proper direction.

Class ω_1 points are given by:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Class ω_2 points are given by:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(d) (5 points)

Given sampled data points, a student found that a linear classifier outperformed the Bayes classifier. Since she correctly assumed that the data was generated by Normal distributions, what probably was the explanation? (The Bayes classifier is supposed to be optimal!)

(e) (5 points)

Find a discriminant function $g(x)$ that successfully classifies the following data points. Class ω_1 points are given by:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Class ω_2 points are given by:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

5. (5 points) EXTRA CREDIT

   Find the a priori probabilities $P(\omega_1)$ and $P(\omega_2)$ that maximizes the Bayes error in Problem 1.